Morse Theory

Goal: Investigating how functions defined on a Mfld are related to its geometric aspects

① Basics
- ↳ critical points
- ↳ degenerated & non-degenerated
- ↳ Morse Lemma

② Morse functions & the two sphere

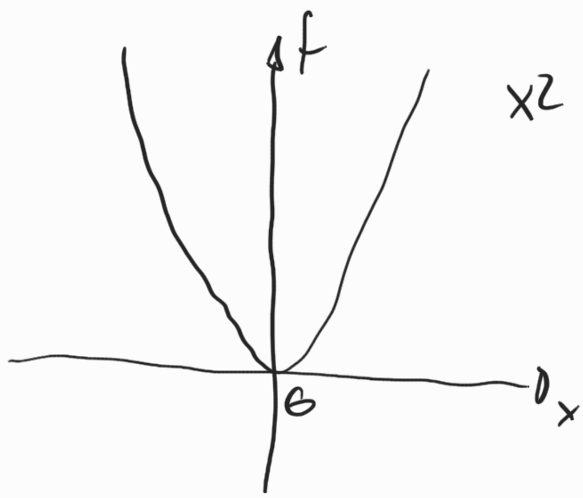
③ Handle decomposition
- compact surfaces
- compact Mflds

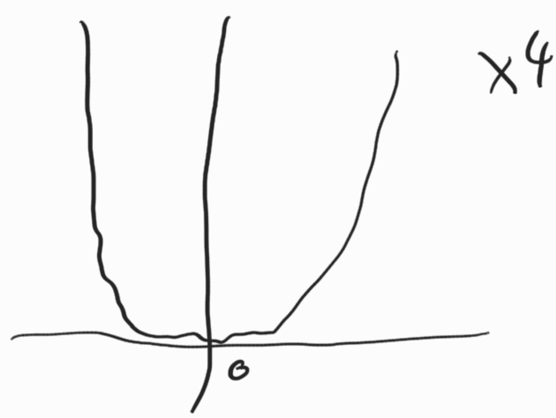
Def. 1.1

Let f be a real valued function.
A point $x_0 \in \mathbb{R}$ w/ $f'(x_0) = 0$ is called a critical point.

x_0 non-deg. if $f''(x_0) \neq 0$

x_0 degenerated if $f''(x_0) = 0$





Under pert.:

- non-deg. critical points are stable
- deg. critical points are unstable

Def. 1.2 The gradial ∇f of a function is the vector field on the domain of f that takes the values $\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ at each point. Donet the ∇f by ∇f and the value at p as $\nabla f|_p = \nabla f(p)$

Def. 1.3

A critical point of $f: \mathbb{R}^n \to \mathbb{R}$ is a point $P \in \mathbb{R}^n$ s.t. $\nabla f(P_0) = 0$.

A critical value of f is a value $c \in \mathbb{R}$ s.t. $f(P_0) = c$.

Def. 1.4

A critical point P_0 of a smooth function f is called degenerated if $\det(H_f(P_0)) = 0$

J_{p_0} the Jacobian of coord. trafo

$$(y_1, \dots, y_n)^T = J_{p_0} (x_1, \dots, x_n)^T$$

$$\tilde{H}_f(p) = J_{p_0}^T \, H_f(p_0) \, J_{p_0}$$

<u>Prop. 1.5</u> The property that p_0 is a degenerated / non-degenerated critical point does not depend on the choice of local coord.

<u>Def 1.6</u> Given a smooth Mfld M and a smooth function $f: M \to \mathbb{R}$, we say that f is <u>Morse</u> if f has <u>no</u> degenerated critical points.

<u>Def. 1.7</u> M be a smooth Mfld.

$f: M \to \mathbb{R}$ smooth and $p_0 \in M$ is a non-degenerated critical point of f.
Then the index of f at p_0 is defined to be the number of negative eigenvalues of the Hessian at p_0.

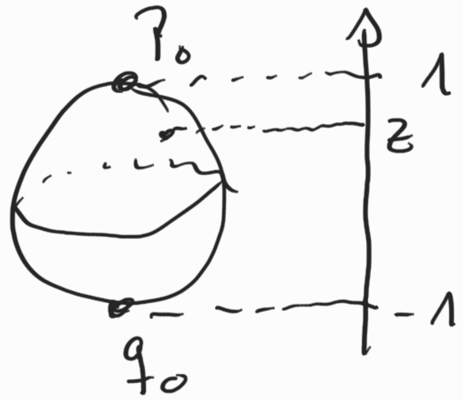
<u>Prop 1.8</u> The index of f at p_0 do not depend on the choice of local coord.

<u>Proof:</u> Sylvester's law $\to$ # negative eigenvalues of $H_f(p)$ is independent of the way it is diagonalized.

$\Rightarrow$ # negative eigenvalues is invariant under local coord. transf.

$\square$

Example: Height function



$$f(x,y) = \pm\sqrt{1-x^2-y^2}$$

2 critical points

$p_0 = (0,0,1)$ $\quad q_0 = (0,0,-1)$

$$H_f(p_0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \neq 0 \qquad \text{index} = 2$$

$$H_f(q_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq 0 \qquad \text{index} = 0$$

Theorem 1.9 (Existence of Morse functions)

Let M be a closed Mfld and $f_0 : M \to \mathbb{R}$ be smooth. Then there exists a Morse function f on M that is an arbitrarily close approx. of f_0.

Sketch
- $\{U_\ell\}_{1\le \ell\le h}$ finite open cover of M.

- For each U_ℓ $\exists$ compact subset K_ℓ of U_ℓ w/ $\{K_\ell\}$ a cover of M by compact sets.

$\Rightarrow$ idea: inductively define function f_ℓ on M s.t. f_ℓ is Morse on $\bigcup_{j=1}^{\ell} K_j =: C_\ell$. When $\ell=h$ we have f_h Morse on $C_h = M$.

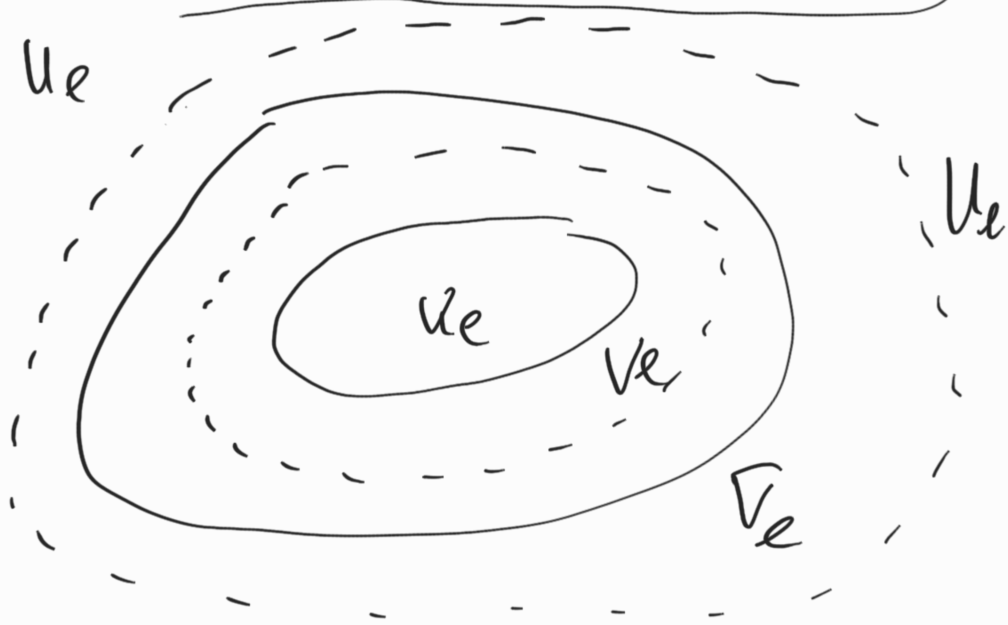
hypothesis: $f_{\ell-1}: M \to \mathbb{R}$ Morse on $C_{\ell-1}$

$\Rightarrow$ exists f_ℓ Morse on $C_{\ell-1} \cup K_\ell$

Do this Lemma which states that if $\{x_1, \dots, x_n\}$ are local coord. on U_ℓ, then exists a real number $\{a_i\}$ s.t.

$$\boxed{f_{\ell-1}(x_1, \dots, x_n) - (a_1x_1 + a_2x_2 + \dots + a_nx_n)}$$

is Morse on U_ℓ



$h_\ell : U_\ell \to [0,1]$

being 1 on

open $V_\ell \subset U_\ell$

h_ℓ is 0 outside $\bar{V}_\ell$

∎

Theorem 1.10 (Morse Lemma)

Let f be a Morse function on a smooth Mfld M an p_0 be a critical point of f. Then there exist local coord. $\{x_1, \dots, x_n\}$ on a nghd U of p st. on U f has the form:

$$f(x_1, \dots, x_n) = -x_1^2 - x_2^2 - \dots - x_n^2 + x_{n+1}^2 + \dots + x_n^2$$

where k is the index of f at p_0.

p corresponds to the origin of this coord. system.

Lemma 1.11

$f: \mathbb{R}^n \to \mathbb{R}$ be smooth on a convex nghbd $U \subset \mathbb{R}^n$ containing the origin and suppose $f(0, \dots, 0) = 0$.

Then there ex. smooth functions $\{g_i\}_{1 \le i \le n}$ defined on U st.

$$f = \sum_{i=1}^{n} x_i g_i$$

w/ $g_i(0, \dots, 0) = \frac{\partial f}{\partial x_i}(0, \dots, 0)$

Proof (Theorem 1.10)

$\{y_1, \dots, y_n\}$ being local coord. on U of p

using Lemma 1.11:

$$g_i : \mathbb{R}^n \to \mathbb{R} \quad \text{st} \quad f(y_1, \dots, y_n) = \sum_{i=1}^{n} y_i g_i(y_1, \dots, y_n)$$

w/ $g_i(0) = \frac{\partial f}{\partial x_i}(0)$

$\Longrightarrow$ apply again on g_i

$$h_{ij} : \mathbb{R}^n \to \mathbb{R} \quad \text{s.t.} \quad h_{ij}(0) = \frac{\partial g_i}{\partial x_j}(0)$$

$$\Rightarrow f(y_1, \dots, y_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} y_j y_i h_{ij}(y_1, \dots, y_n)$$

$$\frac{\partial f^2}{\partial y_i \partial y_j}(p) = \begin{cases} 2h_{ii} & i=j \\ h_{ij} & i \neq j \end{cases} \qquad p = (0, \dots, 0)$$

$\Rightarrow$ Diagonalize Hessian $\to$ λ_i be the ith diagonal entry of $H_f(p)$

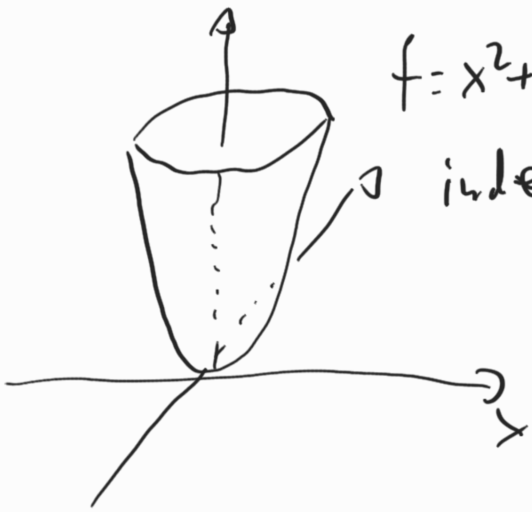
$$\Rightarrow f(\bar{x}_1, \dots, \bar{x}_n) = \sum_{i=1}^{n} \frac{\lambda_i}{2} \bar{x}_i^2$$

$$x_i = \gamma_p(\bar{x}_i) = \text{sign}(\lambda_i) \sqrt{\frac{|\lambda_i|}{2}}\, \bar{x}_i$$

$$\Rightarrow f(x_1, \dots, x_n) = \text{sign}(\lambda_1) x_1^2 + \dots + \text{sign}(\lambda_n) x_n^2$$

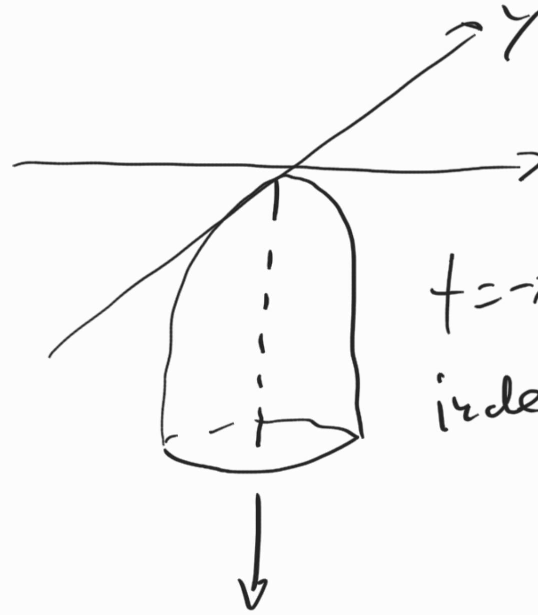
$\square$

Example 2-dim case



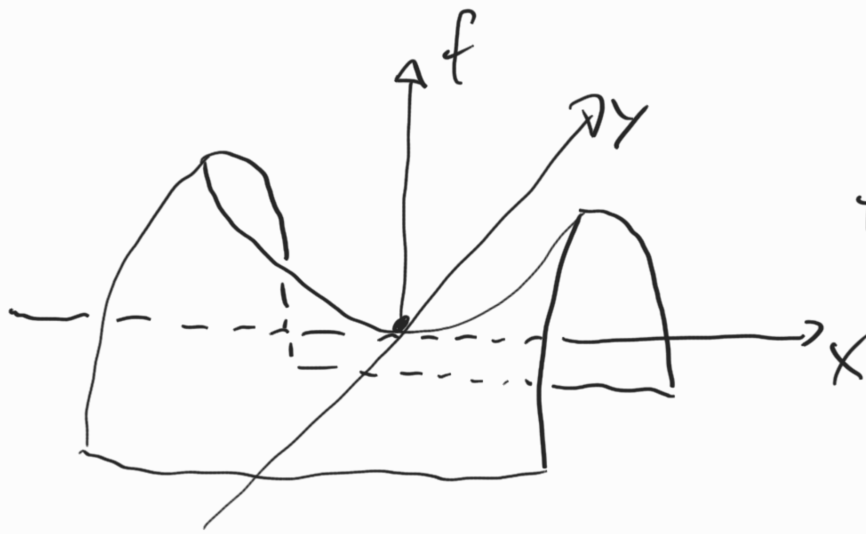
$f = x^2 + y^2$

index $= 0$



$f = -x^2 - y^2$

index $= 2$



$f = -x^2 - y$

index $= 1$

$\Rightarrow$ f must not have any critical point too near to any other.

Corollary 1.12 Non-deg. critical points on any Mfld can be isolated by open nghbds

②

Theorem 2.1

Let M be a closed surface.
Suppose that there exist a Morse function $f: M \to \mathbb{R}$ w/ exactly two critical points. Then M is diffeomorphic to S^2

Note: Generalization to n-dim:

Reeb sphere theorem

$\Rightarrow$ homeomorphic to S^n

Lemma 2.2

Let $f: M \to \mathbb{R}$ be a smooth function which takes constant values on the boundary circles $C(p_0)$ & $C(q_0)$.
Assume f has no critical point M.
Then $M \cong C(q_0) \times [0,1]$

Lemma 2.3

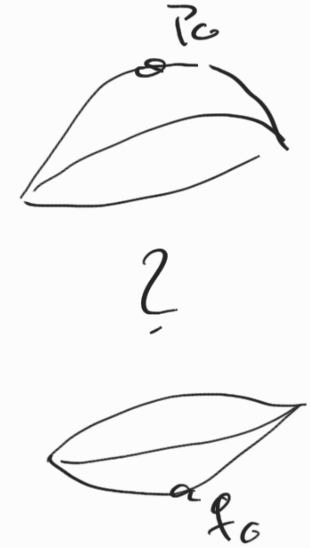
Let $h: \partial D_1 \longrightarrow \partial D_1$ be a diffeom. Then we can extend h to a diffeom. $H: D_1 \longrightarrow D_1$

Proof

f is smooth, M compact $\Rightarrow$ p_0 maximum value

q_0 minimum value

Theorem 1.10 expres f locally:



$$f = \begin{cases} -x^2-y^2+A, & \text{around } p_0 \\ x^2+y^2+a, & \text{around } q_0 \end{cases}$$

$D(p_0) := \{ \text{set of points } p \in M \text{ w/ } A-\varepsilon \leq f(p) \leq A \}$

$$= \{(x,y) \mid x^2+y^2 \leq \varepsilon\} \cong D^2$$

$D(q_0) \cong D^2$

$M_0 := M \setminus (D(p_0) \cup D(q_0))$

Restriction of $f: M_0 \to \mathbb{R}$

takes constant values at

$C(p_0)$ & $C(q_0)$

$\Rightarrow$ use Lemma 2.2. since $C(q_0) \cong S^1$

$$\Rightarrow \quad M_0 \cong S^1 \times [0,1]$$

$N_0 := M_0 \cup D(q_0)$ (along boundary of $D(q_0)$)

$H_1 : N_0 \to D_-$

$h : C(p_0) \to \partial N_0$

$H_1|_{\partial N_0} \circ h : C(p_0) \to \partial D_- \cong \partial D_+$

Lemma 2.3 extends to a diffeo

$H_2 : D(p_0) \to D_+$

$\Rightarrow$ glueing two diffeos together:

$$H : N_0 \cup_h D(p_0) \cong M \longrightarrow D_- \cup D_+ \cong S^2$$

$\boxtimes$

Lemma 2.4

A Morse function $f : M \to \mathbb{R}$ defined on a closed Mfld. M has only finite number of critical points

Contradiction: infinitly many critical points

$\Rightarrow$ seq. of critical points:

$\{q_n\}_{n \geq 1} \subset M$

compactness $\rightarrow$ conv. sub.

$\Rightarrow$ choose further $\{q_{n_k}\}_{k \geq 1} \subset M$

$q_{n_k} \to q_0, k \to \infty$

$f(q_0)$ is also critial point. $\lightning$

③

$f: M \to \mathbb{R}$ Morse function. M closed & connected surface.

Define $M_t := \{p \in M \mid f(p) \leq t\} \subset M$

$L_t := \{p \in M \mid f(p) = t\} \subset M$

$t \leq a \Rightarrow M_t = \emptyset$

$A \leq t \Rightarrow M_t = M$

Lemma 3.1

Let $b < c$ s.t f has no critical point in $[b,c]$. Then M_b & M_c are diffeomorphic

1) index of p_0 is zero

locally form : $f = x^2 + y^2 + c_0$

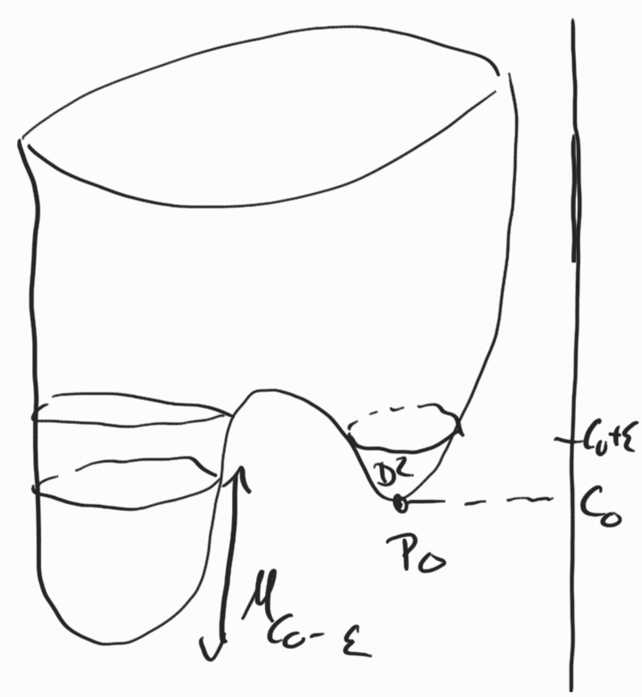
if c_0 minimum $\Rightarrow$ $M_{c_0-\varepsilon} = \emptyset$

$$M_{c_0+\varepsilon} = \{ p \in M \mid f(p) \leq c_0 + \varepsilon \}$$
$$= \{ (x,y) \mid x^2+y^2 \leq \varepsilon \} \cong D^2$$

c_0 not minimum, $M_{c_0-\varepsilon} \neq \emptyset$

$$\Rightarrow M_{c_0+\varepsilon} \cong M_{c_0-\varepsilon} \sqcup D^2$$

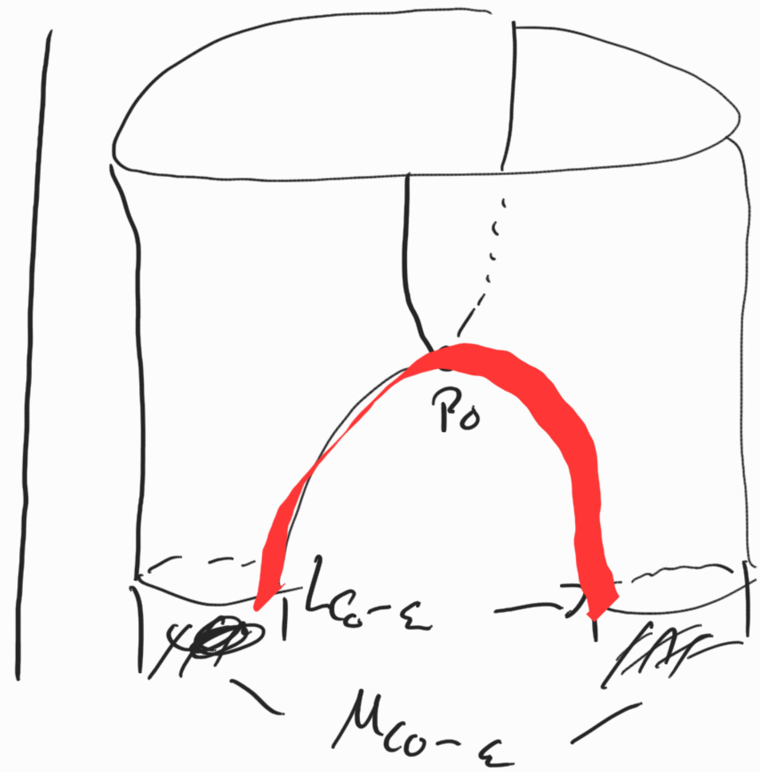
Crossing c_0, a disk pops out and M_t becomes diffeom. to disjoint union of $M_{c_0-\varepsilon} \sqcup D^2$

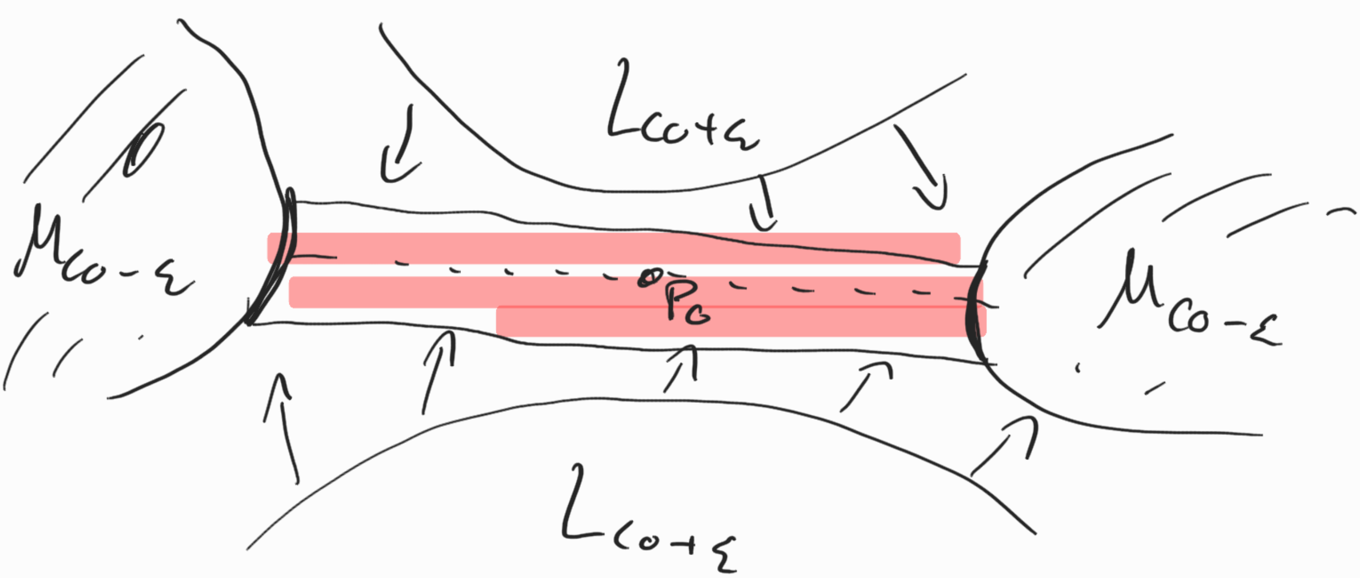


2) <u>Index of p_0 is equal one:</u>

local form: $f = -x^2 + y^2 + c_0$

red line connecting edges $L_{c_0-\varepsilon}$





Diffeom. to rectangle $\Rightarrow$ $\boxed{D_1 \times D_1}$

intersecting w/ $M_{c_0-\varepsilon}$ corresponds to paste $\boxed{\partial D^1 \times D^1}$ $\Rightarrow$ 1-handle attached to $M_{c_0-\varepsilon}$

$$M_{c_0+\varepsilon} \cong M_{c_0-\varepsilon} \cup D^1 \times D^1$$

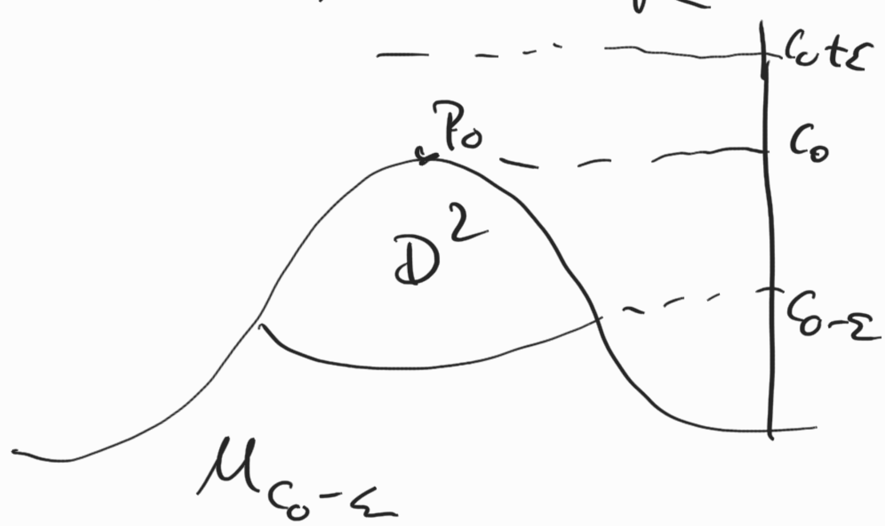
3) Index to p_0 is equal two

local form: $f = -x^2 - y^2 + c_0$

$\Rightarrow$ $M_{c_0-\varepsilon} = \{ (x,y) \mid x^2+y^2 \geq \varepsilon \}$

$M_{c_0-\varepsilon}$ outside a disk w/ radius $\sqrt{\varepsilon}$

Bowl D^2 capping $M_{c_0-\varepsilon}$ from above $\Rightarrow$ 2-handle



$$\boxed{M_{c_0+\varepsilon} \cong M_{c_0-\varepsilon} \cup D^2}$$

Lemma 3.2

Let M be a smooth mfld w/ f is a Morse function on M. then if p & q are both critical points of f s.t $f(p) = f(q)$, then there exists a smooth mfld $M' \cong M$ s.t. $f(p) \neq f(q)$

theorem 3.3

A closed surface M admits a Morse function $f: M \to \mathbb{R}$ and therefore M can be described as a union of finitely many 0,1,2 handles.

Proof:

M compact $\Rightarrow$ th. 1.8 exists at a Morse function.

$A, B \in \mathbb{R}$ s.t $M_A = \{\emptyset\}$, $M_B = M$

Compactness guarantees $\rightarrow$ finitely many critical points. Lemma 2.4.

Lemma 3.2 says that we can adjust M by diffeo. s.t. $f(p_i) \neq f(p_j)$ $i \neq j$

$L := \{\#$ critical points$\}$

Index each critical point s.t. if $i < j$

$\Rightarrow f(p_i) < f(p_j)$

P_1 lowest critical point & P_L highest.

Define $a_i := \dfrac{f(p_i) + f(p_{i+1})}{2}$ $\quad i \in \{1, \ldots, L-1\}$

$\Rightarrow a_i$ defined in that way s.t.

sublevel sets M_{a_i} containing only critical points up to p_i.

Set $M_0 = \{\emptyset\}$ $\quad M_L = M$

We see that $f^{-1}[a_i, a_{i+1}]$ contains exactly one critical point.

$M_{a_{i+1}}$ is diffeomorphic to M_{a_i} attached to h-handle (0,1,2) where is the index of a_{i+1}

Furthermore Lemma 3.1. we have that the topology of the two sublevel sets M_{a_i} and M_b ($b > a_i$) only differ when $b > f(p_{i+1})$

Therefore, the sequence $\{M_0, M_1, \ldots, M_{L-1}, M_L\}$ is a handle decomposition of M $\square$

Theorem 3.4

Let M be a compact smooth Mfld.
and $f: M \to \mathbb{R}$ be a Morse function.
Suppose $a, b \in \mathbb{R}$ st $f^{-1}[a,b]$ non-empty
If $f^{-1}[a,b]$ contains one critical point
of f w/ index k, then M_b is
diffeomorphic to the union of M_a with a
k-handle.

Theorem 3.5

There exists a handle decomposition
for every compact smooth Mfld.